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Branching Semigroups with a Compatible Set-Valued Operator

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ABSTRACT

This paper introduces a branching semigroup, defined as a semigroup equipped with an extensive set-valued unary operator that is compatible with semigroup multiplication. The construction separates deterministic composition from nondeterministic branching while retaining enough algebraic structure to support ideals, homomorphisms, quotients, and graph-based analysis. We prove that ordinary semigroups embed as singleton-branching structures, intersections of branch ideals are branch ideals, strong homomorphic images inherit the branching structure, and kernels produce quotient branching semigroups satisfying a first isomorphism theorem. For finite structures, ascending and descending chains of branch ideals stabilize. We associate a directed branch graph to every branching semigroup and obtain a canonical decomposition into strongly connected components; sink components are branch-closed, although components need not be subsemigroups without an additional closure assumption. Two fully verified examples are given: a coset-based structure on the cyclic group of order four and a truncated-addition monoid with an explicit chain of branch ideals. Potential uses in nondeterministic composition, routing, and decision systems are presented as modeling interpretations rather than empirical claims.

1. Introduction

Algebraic structures usually encode deterministic composition: a pair of elements produces a single result. Many computational and networked systems, however, combine such composition with a family of admissible alternatives, refinements, or successors. One way to model this situation is to retain a single-valued associative operation and add a separate set-valued operator that records the available branches.

Semigroup theory provides the natural base for associative composition [1]. Universal algebra supplies the general language of operations, homomorphisms, congruences, and quotients [2]. Set-valued algebraic operations are also studied in hyperstructure and multialgebra theory [3-5]. The construction considered here is related to those areas but is deliberately narrower: multiplication remains single-valued, whereas branching is represented by a set-valued unary operator.

The term branch algebra is used in several unrelated specialist settings. Accordingly, the present paper does not assume a pre-existing general theory under that name. It proposes the term branching semigroup for the precise structure defined below. Any claim of novelty beyond this formulation should be evaluated through a dedicated bibliographic search.

The contributions are as follows. First, we give complete definitions of branching semigroups, branch ideals, strong branch homomorphisms, and branch congruences. Second, we establish elementary image, quotient, and ideal results. Third, we associate a directed graph with the branch operator and distinguish graph decomposition from algebraic decomposition. Finally, we present examples in which every axiom can be verified explicitly. The results are foundational. They do not demonstrate empirical performance in artificial intelligence, communication networks, or cryptography. Instead, the final section explains how the axioms may serve as modeling constraints in such domains.

2. Definitions and notation

Let S be a nonempty set. Write $P^*(S)$ for the collection of all nonempty subsets of S. If $(S, \cdot)$ is a semigroup and $A, B \in P^*(S)$, define the setwise product

$$AB = \{ab : a \in A \text{ and } b \in B\}. \quad (2.1)$$

Definition 2.1 (Branching semigroup). A branching semigroup is a triple $B = (S, \cdot, \beta)$, where $(S, \cdot)$ is a semigroup and $\beta: S \to P^*(S)$ satisfies, for all $x, y \in S$:

(B1) $x \in \beta(x)$ (extensivity), and

(B2) $\beta(xy) = \beta(x)\beta(y)$ (multiplicativity).

Condition (B1) says that the original state is always one of its own branches. Condition (B2) says that branching commutes with composition: branching after a product gives exactly the products of branches of the two factors.

The operator β extends to nonempty subsets $A \subseteq S$ by

$$\hat{\beta}(A) = \bigcup\{\beta(a) : a \in A\}. \quad (2.2)$$

Definition 2.2 (Branch subsemigroup). A nonempty subset $T \subseteq S$ is a branch subsemigroup if $TT \subseteq T$ and $\hat{\beta}(T) \subseteq T$. The restricted triple $(T, \cdot, \beta|T)$ is then a branching semigroup.

Definition 2.3 (Branch ideal). A nonempty subset $I \subseteq S$ is a branch ideal if $SI \subseteq I$, $IS \subseteq I$, and $\hat{\beta}(I) \subseteq I$. Thus a branch ideal is a two-sided semigroup ideal that is closed under branching.

Definition 2.4 (Strong branch homomorphism). Let $B = (S, \cdot, \beta)$ and $C = (T, *, \gamma)$ be branching semigroups. A map $f: S \to T$ is a strong branch homomorphism if $f(xy) = f(x)*f(y)$ and $f[\beta(x)] = \gamma(f(x))$ for every $x \in S$.

The equality in Definition 2.4 is stronger than the inclusion $f[\beta(x)] \subseteq \gamma(f(x))$. Equality is used here because it guarantees that the image is closed under every branch available in the codomain.

Definition 2.5 (Branch congruence). A branch congruence on $B = (S, \cdot, \beta)$ is a semigroup congruence θ such that $q\theta[\beta(x)] = q\theta[\beta(y)]$ whenever $x\ \theta\ y$, where $q\theta\colon S \to S/\theta$ is the quotient map.

3. Elementary properties and branch ideals

Proposition 3.1 (Singleton embedding). Every semigroup $(S, \cdot)$ determines a branching semigroup $(S, \cdot, \beta_0)$ by $\beta_0(x) = \{x\}$.

Proof. Extensivity is immediate. Moreover, $\beta_0(xy) = \{xy\} = \{x\}\{y\} = \beta_0(x)\beta_0(y)$. □

Proposition 3.2 (Setwise multiplicativity). For all nonempty $A, B \subseteq S$, $\hat{\beta}(AB) = \hat{\beta}(A)\hat{\beta}(B)$.

Proof. Using Definition 2.1, $\hat{\beta}(AB) = \bigcup_{(a\in A,\ b\in B)} \beta(ab) = \bigcup_{(a\in A,\ b\in B)} \beta(a)\beta(b)$, which is exactly $\hat{\beta}(A)\hat{\beta}(B)$. □

Proposition 3.3 (Intersection of branch ideals). If I and J are branch ideals of B, then $I \cap J$ is a branch ideal.

Proof. Choose $i \in I$ and $j \in J$. Because I is a right ideal, $ij \in I$, and because J is a left ideal, $ij \in J$; hence $I \cap J$ is nonempty. Also $S(I \cap J) \subseteq SI \cap SJ \subseteq I \cap J$ and $(I \cap J)S \subseteq IS \cap JS \subseteq I \cap J$. Finally, if $x \in I \cap J$, then $\beta(x) \subseteq I$ and $\beta(x) \subseteq J$, so $\beta(x) \subseteq I \cap J$. □

Corollary 3.4 (Generated branch ideal). For every nonempty $A \subseteq S$, there is a least branch ideal containing A, namely the intersection of all branch ideals that contain A.

Proof. The family is nonempty because S itself is a branch ideal. Every member contains A, so the intersection is nonempty, and repeated application of Proposition 3.3 gives the result for finite intersections; the same closure argument applies directly to the full intersection. □

Remark 3.5. Extensivity implies $I \subseteq \hat{\beta}(I)$. Therefore the branch-closure condition $\hat{\beta}(I) \subseteq I$ is equivalent to $\hat{\beta}(I) = I$.

4. Homomorphisms and quotient structures

Theorem 4.1 (Strong homomorphic image). If $f: B \to C$ is a strong branch homomorphism, then $f(S)$ is a branch subsemigroup of C.

Proof. The set $f(S)$ is a subsemigroup because f is a semigroup homomorphism. If $y = f(x) \in f(S)$, then $\gamma(y) = \gamma(f(x)) = f[\beta(x)] \subseteq f(S)$. Hence $f(S)$ is closed under the branch operator. □

Proposition 4.2 (Preimages of ideals). Let $f: B \to C$ be a strong branch homomorphism and let J be a branch ideal of C. If $f^{-1}(J)$ is nonempty - in particular, if f is surjective - then $f^{-1}(J)$ is a branch ideal of B.

Proof. The ordinary preimage argument gives $Sf^{-1}(J) \subseteq f^{-1}(J)$ and $f^{-1}(J)S \subseteq f^{-1}(J)$. If $x \in f^{-1}(J)$ and $z \in \beta(x)$, then $f(z) \in f[\beta(x)] = \gamma(f(x)) \subseteq J$. Thus $z \in f^{-1}(J)$. □

Theorem 4.3 (Quotient construction). If θ is a branch congruence on B, define $\beta\theta([x]\theta) = \{[u]\theta : u \in \beta(x)\}$. Then $B/\theta = (S/\theta, \cdot, \beta\theta)$ is a branching semigroup.

Proof. The compatibility condition in Definition 2.5 makes $\beta\theta$ well defined. Extensivity follows from $x \in \beta(x)$. For multiplicativity, $\beta\theta([x][y]) = q\theta[\beta(xy)] = q\theta[\beta(x)\beta(y)] = q\theta[\beta(x)]q\theta[\beta(y)] = \beta\theta([x])\beta\theta([y])$. □

Theorem 4.4 (First isomorphism theorem). Let $f: B \to C$ be a strong branch homomorphism and let ker f be the equivalence relation $x \sim y$ if $f(x) = f(y)$. Then ker f is a branch congruence and B/ker f is isomorphic, as a branching semigroup, to $f(S)$.

Proof. The kernel of a semigroup homomorphism is a semigroup congruence. If $f(x) = f(y)$, then $f[\beta(x)] = \gamma(f(x)) = \gamma(f(y)) = f[\beta(y)]$, so the sets of kernel classes represented by $\beta(x)$ and $\beta(y)$ coincide. Thus ker f is a branch congruence. The usual map $\varphi([x]) = f(x)$ is a bijective semigroup homomorphism onto $f(S)$, and $\varphi[\beta\ker f([x])] = f[\beta(x)] = \gamma(f(x))$; hence φ is a branching-semigroup isomorphism.

5. Finite branch-ideal conditions

Theorem 5.1 (Finite chain condition). If $|S| = n < \infty$, every ascending or descending chain of branch ideals stabilizes. A strictly ascending or strictly descending chain has at most n distinct terms.

Proof. A strict inclusion between finite subsets changes cardinality by at least one. Since every branch ideal is a nonempty subset of the n-element set S, no strict chain can contain more than n members. □

Corollary 5.2. If a finite branching semigroup has a proper branch ideal, then it has a maximal proper branch ideal.

Proof. The finite set of proper branch ideals is a finite partially ordered set under inclusion, so every element lies below a maximal element.

Theorem 5.1 is a finiteness observation, not a structure theorem. For infinite branching semigroups, ascending or descending chains may fail to stabilize.

6. The branch digraph and finite decomposition

Definition 6.1 (Branch digraph). The branch digraph Dβ has vertex set S and a directed edge $x \to y$ precisely when $y \in \beta(x)$. Because $x \in \beta(x)$, every vertex has a loop.

Write $x \rightsquigarrow y$ **if Dβ contains a directed path from x to y. Mutual reachability, defined by** $x \equiv_\beta y$ when $x \rightsquigarrow y$ and $y \rightsquigarrow x$, is an equivalence relation. Its classes are the strongly connected components of Dβ [6,7].

Theorem 6.2 (Canonical graph decomposition). The strongly connected components of Dβ partition S. Contracting each component to one vertex produces an acyclic directed graph, called the condensation digraph.

Proof. Mutual reachability is reflexive, symmetric by definition, and transitive by concatenating paths, so it is an equivalence relation and its classes partition S. A directed cycle among distinct condensation vertices would make the corresponding components mutually reachable, contradicting their maximality. □

Proposition 6.3 (Sink components). If S is finite, the condensation digraph has at least one sink component. **Every sink component C satisfies** $\hat{\beta}(C) \subseteq C$. If, in addition, $CC \subseteq C$, then C is a branch subsemigroup.

Proof. Every finite acyclic digraph has a sink. If C is a sink and $x \in C$, then an edge $x \to y$ cannot leave C, so $\beta(x) \subseteq C$. Hence $\hat{\beta}(C) \subseteq C$. Product closure gives the final assertion. □

Proposition 6.4 (Functoriality of components). A strong branch homomorphism maps each strongly connected component of the domain into a single strongly connected component of the codomain.

Proof. If $x \to y$, then $f(y) \in f[\beta(x)] = \gamma(f(x))$, so $f(x) \to f(y)$. Thus f maps directed paths to directed paths. Mutual reachability is therefore preserved. □

Remark 6.5. A strongly connected component need not be closed under multiplication or under outgoing branches. Consequently, the graph decomposition must not be described as a union of algebraic components unless those closure properties are separately proved.

7. Examples

7.1 Coset branching on the cyclic group of order four

Let $S = Z_4$ under addition modulo 4 and let $H = \{0, 2\}$. Since H is a normal subgroup, define

$\beta(x) = x + H.$

Explicitly, $\beta(0) = \beta(2) = \{0, 2\}$ and $\beta(1) = \beta(3) = \{1, 3\}$. Because $0 \in H$, $x \in x + H$. Moreover,

$\beta(x + y) = x + y + H = x + y + H + H = \beta(x) + \beta(y).$

Hence $(Z_4, +, \beta)$ is a branching semigroup. The branch digraph has exactly two strongly connected components, H and 1 + H. Since a group has no proper nonempty two-sided semigroup ideals, the only branch ideal is Z_4 itself.

x	$\beta(x)$
0	{0, 2}
1	{1, 3}
2	{0, 2}
3	{1, 3}

7.2 Truncated addition and a complete ideal chain

Fix $n \geq 1$ and let $T_n = \{0, 1, \ldots, n\}$. Define $x \odot y = \min(x + y, n)$ and $\beta(k) = \{k, k + 1, \ldots, n\}$. Truncated addition is associative because

$$\min(\min(x + y, n) + z, n) = \min(x + y + z, n) = \min(x + \min(y + z, n), n).$$

Extensivity is immediate. For multiplicativity,

$$\beta(x) \odot \beta(y) = \{r \in T_n : r \geq \min(x + y, n)\} = \beta(x \odot y).$$

For each k, let $I_k = \{k, k + 1, \ldots, n\}$. Then $T_n \odot I_k \subseteq I_k$, $I_k \odot T_n \subseteq I_k$, and $\hat{\beta}(I_k) = I_k$, so I_k is a branch ideal. Conversely, if I is any branch ideal and $k = \min I$, then $\beta(k) = I_k \subseteq I$ while $I \subseteq I_k$; hence $I = I_k$. Therefore the complete branch-ideal lattice is the chain

$$\{n\} = I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1 \subsetneq I_0 = T_n.$$

The branch digraph has edges $k \to r$ whenever $r \geq k$. Its strongly connected components are the singletons, and $\{n\}$ is the unique sink component.

8. Modeling interpretations

8.1 Nondeterministic composition

Suppose semigroup elements represent composable actions and $\beta(x)$ is the set of admissible implementations or refinements of action x. The axiom $\beta(xy) = \beta(x)\beta(y)$ states that every implementation of a composite action is obtained by composing an implementation of x with an implementation of y, and conversely. This is a strong compositionality requirement; systems with only one direction of inclusion should use a weaker variant.

8.2 Routing and path systems

For routes with compatible endpoints, composition is concatenation and β may list permitted alternatives that preserve the endpoints. In a general network, concatenation is only partially defined, so a category or semigroupoid version of the theory is more natural than an ordinary semigroup. The present axioms can be applied directly only after routes are encoded in a total semigroup, for example by adjoining a failure element.

8.3 Decision and information systems

An element may encode a decision rule, a state transformation, or a query plan, while β lists acceptable alternatives. Branch ideals can then represent absorbing collections of states or plans that remain closed under both context composition and permitted alternatives. This interpretation is conceptual; a practical study would need a domain-specific encoding, data, and an evaluation criterion.

8.4 Finite verification

A finite candidate structure can be checked from a multiplication table and a branch-incidence matrix. Associativity requires $O(n^3)$ table comparisons. Extensivity requires checking n diagonal incidences. Multiplicativity can be tested by comparing β(xy) with the setwise product β(x)β(y) for every pair (x, y); a direct implementation is polynomial and can be accelerated with bit-set representations. These checks should accompany every computational example.

9. Relation to existing algebraic frameworks

In a multialgebra, one or more basic operations return nonempty sets rather than individual elements [4,5]. A semihypergroup similarly replaces binary multiplication with a hyperoperation [3]. In contrast, a branching semigroup retains an ordinary semigroup multiplication and adds one set-valued unary operator.

The distinction is useful when deterministic composition and nondeterministic variation play different semantic roles. Nevertheless, the power-set extension $\hat{\beta}$ and the use of setwise products place the construction close to hyperstructure theory. Results should therefore be compared with that literature before originality is claimed.

Several natural variants remain open for study: replacing equality in (B2) by one of the inclusions $\beta(xy) \subseteq \beta(x)\beta(y)$ or $\beta(x)\beta(y) \subseteq \beta(xy)$; adding idempotence $\hat{\beta}(\beta(x)) = \beta(x)$; imposing an order or topology; studying one-sided branch ideals; and extending the framework from semigroups to categories, partial algebras, or typed transition systems.

10. Conclusion

A precise set-valued extension of semigroups has been developed. The branching operator is extensive and multiplicative, which makes it possible to define branch ideals and strong branch homomorphisms without ambiguity. The basic theory includes closure under intersections, homomorphic images, quotient structures, a first isomorphism theorem, finite chain conditions, and a canonical graph decomposition. The examples show that the axioms are nonvacuous and that branch ideals can have a transparent structure. The graph decomposition should not be confused with an algebraic direct-sum or component decomposition: additional closure assumptions are required. Future work should focus on weaker compatibility axioms, nontrivial representation results, categorical generalizations, and fully specified applications.

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