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Some Studies in Differential Geometry: Gradient Almost Ricci Solitons on Rotational Warped Surfaces

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ABSTRACT

This paper studies gradient almost Ricci soliton structures on two-dimensional rotational warped manifolds. Let $M = I \times S^1$ be endowed with the metric $g = dr^2 + \varphi(r)^2\, d\theta^2$, where I is an interval and φ is a positive smooth warping function. For radial potential functions $f = f(r)$, we derive a complete reduction of the gradient almost Ricci soliton equation $\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g$ to an ordinary differential system. In particular, we prove that every radial solution satisfies $f'(r) = \mu\varphi(r)$ for a constant μ, and that the soliton function is $\lambda(r) = -\varphi''(r)/\varphi(r) + \mu\varphi'(r)$. Consequently, the potential vector field ∇f is necessarily conformal. We further establish integral identities for smooth sphere-type rotational metrics, describe the relation between rotational Killing fields and soliton potentials, and compute explicit examples on the Euclidean plane, cylinder, sphere, hyperbolic plane and catenoid-type surfaces. The results provide a transparent two-dimensional model linking warped geometry, curvature, conformal vector fields and almost Ricci soliton structures.

Introduction

Differential geometry investigates smooth manifolds through geometric quantities such as metrics, connections, curvature tensors, vector fields and geometric flows. Among recent topics of considerable interest are Ricci solitons and almost Ricci solitons, which generalise Einstein metrics and arise naturally in the study of Ricci flow.

The editorial by Blaga surveys a Special Issue on differentiable manifolds and geometric structures containing work on curvature restrictions, geometric solitons, Killing and projective vector fields, Sasakian-type structures, warped products and submanifold inequalities [1]. In particular, Blaga and Deshmukh studied potential vector fields of almost Ricci solitons on compact Riemannian manifolds [2], while Blaga and Özgür examined Killing and 2-Killing fields on doubly warped products and their connections with Ricci soliton geometry [3]. These themes indicate that the interaction between symmetry fields, curvature and soliton equations remains an active direction in differential geometry.

Warped products provide an especially useful framework for constructing and analysing soliton metrics. Feitosa, Freitas Filho, Gomes and Pina derived necessary and sufficient conditions for gradient Ricci almost solitons realised as warped products and presented explicit construction methods [4]. The present paper develops a concrete two-dimensional specialisation: rotational warped surfaces with radial potential functions. This setting is mathematically tractable, geometrically visual and directly connected with classical surfaces of revolution.

The principal objective is to derive, prove and illustrate the radial gradient almost Ricci soliton equation on the metric $g = dr^2 + \varphi(r)^2\, d\theta^2$.

The main conclusions are as follows.

1. A radial gradient almost Ricci soliton exists precisely when its potential satisfies $f' = \mu\varphi$ for some constant μ.

2. Its soliton function is explicitly determined by curvature and the derivative of the warping function: $\lambda = K + \mu\varphi' = -\varphi''/\varphi + \mu\varphi'$.
3. The potential vector field is conformal: $\mathscr{L}_{\nabla f}\, g = 2\mu\varphi' g$.
4. For smooth rotational metrics on the two-sphere, the average value of λ is determined entirely by the area: $\int_M \lambda \, dA = 4\pi$.
5. Standard geometries, including the plane, cylinder, sphere and hyperbolic plane, arise as explicit examples.

The paper is organised as follows. Section 2 recalls the required geometric definitions. Section 3 computes the connection and Gaussian curvature of rotational warped surfaces. Section 4 establishes the main classification theorem for radial gradient almost Ricci solitons. Section 5 examines symmetry vector fields. Section 6 proves global integral identities on sphere-type metrics. Section 7 presents explicit worked examples. Section 8 describes figures suitable for the submitted manuscript. Section 9 discusses the contribution and its limitations.

2. Basic Definitions

2.1 Riemannian manifolds and curvature

Let M be a smooth two-dimensional manifold and let g be a Riemannian metric on M. The Levi-Civita connection of g is denoted by ∇. It is the unique connection satisfying $\nabla g = 0$ and $\nabla_X Y - \nabla_Y X = [X,Y]$ for all smooth vector fields X and Y.

The curvature tensor is defined by $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z$.

On a two-dimensional Riemannian manifold, the Ricci tensor is completely determined by the Gaussian curvature K: $Ric = Kg$.

The scalar curvature is therefore $S = 2K$.

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2.2 Gradient almost Ricci solitons

A gradient almost Ricci soliton on a Riemannian manifold (M,g) is a pair of smooth functions f and λ satisfying $\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g$, (2.1)

where Hess f is the Hessian tensor of f: $\mathrm{Hess}\, f(X,Y) = g(\nabla_X \nabla f, Y)$.

The function f is called the potential function, ∇f is the potential vector field, and λ is the soliton function.

When λ is constant, equation (2.1) becomes the gradient Ricci soliton equation. When λ is allowed to vary on M, the structure is an almost Ricci soliton.

A gradient Ricci soliton with constant λ is commonly classified as shrinking, steady or expanding according to the sign convention adopted for λ. In this paper, no classification terminology is needed; only equation (2.1) is used.

2.3 Conformal and Killing vector fields

A vector field V is Killing when $\mathscr{L}_V g = 0$, where $\mathscr{L}$ denotes the Lie derivative.

A vector field V is conformal when there exists a smooth function ρ such that $\mathscr{L}_V g = 2\rho g$.

For a gradient vector field $V = \nabla f$, one has $\mathscr{L}_{\nabla f} g = 2\, \mathrm{Hess}\, f$. (2.2)

Thus, the conformality of ∇f is equivalent to the Hessian being proportional to the metric.

3. Rotational Warped Surfaces

Let $I \subset \mathbb{R}$ be an interval and let $\varphi : I \to (0,\infty)$ be smooth. Consider $M = I \times S^1$ with coordinates (r,θ) and metric $g = dr^2 + \varphi(r)^2\, d\theta^2$. (3.1)

Geometrically, r measures distance along meridian curves and θ is the angular coordinate. Metrics of the form (3.1) include local models of planes, cylinders, spheres, hyperbolic surfaces and many surfaces of revolution.

Lemma 3.1. Levi-Civita connection

For the metric (3.1), the non-zero Christoffel symbols are $\Gamma^r\theta\theta = -\varphi\varphi'$, $\Gamma\theta r\theta = \Gamma\theta\theta r = \varphi'/\varphi$.

Equivalently, $\nabla\partial r\ \partial r = 0$, (3.2)

$\nabla\partial r\ \partial\theta = \nabla\partial\theta\ \partial r = (\varphi'/\varphi)\partial\theta$, (3.3)

$\nabla\partial\theta\ \partial\theta = -\varphi\varphi'\partial r$. (3.4)

Proof

The coordinate components of the metric are

$g^{rr} = 1$, $g\theta\theta = \varphi^2$, $g^r\theta = 0$.

The inverse metric satisfies

$g^{rr} = 1$, $g\theta\theta = 1/\varphi^2$.

Using

$\Gamma^k ij = \frac{1}{2} g^k\ell(\partial i\ gj\ell + \partial j\ gi\ell - \partial\ell\ gij)$,

and observing that the only non-constant metric component is $g\theta\theta = \varphi^2$, we obtain

$\Gamma^r\theta\theta = -\frac{1}{2}\partial r(\varphi^2) = -\varphi\varphi'$,

and

$\Gamma\theta r\theta = \Gamma\theta\theta r = \frac{1}{2}(1/\varphi^2)\partial r(\varphi^2) = \varphi'/\varphi$.

All other Christoffel symbols vanish. This proves the formulas. ■

Proposition 3.2. Gaussian curvature

The Gaussian curvature of the rotational warped metric (3.1) is

$K = -\varphi''/\varphi$. (3.5)

Consequently,

$\mathrm{Ric} = -(\varphi''/\varphi)g$ (3.6)

and

$S = -2\varphi''/\varphi$. (3.7)

Proof

Using Lemma 3.1, compute

$R(\partial r,\partial\theta)\partial\theta$
$= \nabla\partial r(-\varphi\varphi'\partial r) - \nabla\partial\theta((\varphi'/\varphi)\partial\theta)$
$= -(\varphi'^2 + \varphi\varphi'')\partial r + \varphi'^2\partial r$
$= -\varphi\varphi''\partial r$.

Therefore,

$g(R(\partial r,\partial\theta)\partial\theta,\partial r) = -\varphi\varphi''$.

Since

$g(\partial r,\partial r)g(\partial\theta,\partial\theta) - g(\partial r,\partial\theta)^2 = \varphi^2$,

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the sectional curvature is

$K = -\varphi\varphi''/\varphi^2 = -\varphi''/\varphi.$

Because M is two-dimensional, Ric = Kg and S = 2K. ■

4. Radial Gradient Almost Ricci Solitons

Assume that the potential function depends only on r:

$f = f(r).$

Then

$\nabla f = f'(r)\partial r.$ (4.1)

Lemma 4.1. Hessian of a radial function

For f = f(r), the Hessian components are

$\mathrm{Hess}\, f(\partial r,\partial r) = f''$, (4.2)

$\mathrm{Hess}\, f(\partial r,\partial\theta) = 0$, (4.3)

$\mathrm{Hess}\, f(\partial\theta,\partial\theta) = \varphi\varphi' f'$. (4.4)

Proof

Since $\nabla f = f'\partial r$,

$\nabla_{\partial r} \nabla f = f''\partial r$,

and therefore

$\mathrm{Hess}\, f(\partial r,\partial r) = g(f''\partial r,\partial r) = f''.$

Similarly,

$\nabla_{\partial\theta} \nabla f = f' \nabla_{\partial\theta} \partial r = f'(\varphi'/\varphi)\partial\theta.$

Hence,

Hess $f(\partial\theta,\partial\theta)$
$= g(f'(\varphi'/\varphi)\partial\theta,\partial\theta)$
$= f'(\varphi'/\varphi)\varphi^2$
$= \varphi\varphi' f'.$

The mixed term vanishes because ∂r and $\partial\theta$ are orthogonal. ■

Theorem 4.2. Classification of radial gradient almost Ricci solitons

Let $M = I \times S^1$ be equipped with

$g = dr^2 + \varphi(r)^2 d\theta^2,$

where $\varphi > 0$ is smooth. Let $f = f(r)$. Then (M,g,f,λ) is a gradient almost Ricci soliton if and only if there exist constants μ and b such that

$f(r) = \mu \int \varphi(r)\, dr + b,$ (4.5)

and

$\lambda(r) = -\varphi''(r)/\varphi(r) + \mu\varphi'(r).$ (4.6)

Equivalently,

$f'(r) = \mu\varphi(r)$ (4.7)

and

$\lambda = K + \mu\varphi'.$ (4.8)

Proof

Since Ric = Kg, with $K = -\varphi''/\varphi$, the soliton equation

$$\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g$$

evaluated on $(\partial r, \partial r)$ gives

$$K + f'' = \lambda. \quad (4.9)$$

Evaluating the same equation on $(\partial\theta, \partial\theta)$ gives

$$K\varphi^2 + \varphi\varphi' f' = \lambda\varphi^2.$$

Since $\varphi > 0$, division by φ^2 yields

$$K + (\varphi'/\varphi)f' = \lambda. \quad (4.10)$$

Subtracting equation (4.10) from equation (4.9), we obtain

$$f'' - (\varphi'/\varphi)f' = 0.$$

This is equivalent to

$$d/dr\, (f'/\varphi) = 0.$$

Therefore,

$$f'/\varphi = \mu$$

for some constant μ, and thus

$$f' = \mu\varphi.$$

After integration,

$f(r) = \mu \int \varphi(r)\, dr + b,$

where b is constant.

Substituting $f' = \mu\varphi$ into equation (4.10) gives

$\lambda = K + \mu\varphi'.$

Using $K = -\varphi''/\varphi$, we obtain

$\lambda = -\varphi''/\varphi + \mu\varphi'.$

Conversely, assume that f and λ satisfy (4.5) and (4.6). Then

$f' = \mu\varphi, \ f'' = \mu\varphi'.$

Hence

$\text{Hess } f(\partial r,\partial r) = \mu\varphi',$

and

$\text{Hess } f(\partial\theta,\partial\theta) = \varphi\varphi'(\mu\varphi) = \mu\varphi'\varphi^2.$

Thus,

$\text{Hess } f = \mu\varphi' g.$

Since $\text{Ric} = Kg$,

$\text{Ric} + \text{Hess } f = (K + \mu\varphi')g = \lambda g.$

Therefore, (M,g,f,λ) is a gradient almost Ricci soliton.

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Corollary 4.3. Conformality of the potential vector field

Under the hypotheses of Theorem 4.2, the potential vector field satisfies

$\nabla f = \mu\varphi\partial r$ (4.11)

and

$\mathscr{L}_{\nabla f}\, g = 2\mu\varphi' g$. (4.12)

Therefore, every radial gradient almost Ricci soliton on a rotational warped surface has a conformal potential vector field.

Proof

From Theorem 4.2,

$f' = \mu\varphi$.

By Lemma 4.1,

$\mathrm{Hess}\, f = \mu\varphi' g$.

Using identity (2.2),

$\mathscr{L}_{\nabla f}\, g = 2\, \mathrm{Hess}\, f = 2\mu\varphi' g$.

Thus ∇f is conformal. ■

Corollary 4.4. Criterion for an ordinary gradient Ricci soliton

The radial almost Ricci soliton of Theorem 4.2 is an ordinary gradient Ricci soliton precisely when

$-\varphi''/\varphi + \mu\varphi' = \text{constant}$. (4.13)

Equivalently, for a constant λ,

$\varphi'' - \mu\varphi\varphi' + \lambda\varphi = 0$. (4.14)

Proof

An almost Ricci soliton is an ordinary Ricci soliton exactly when λ is constant. Formula (4.6) gives condition (4.13). Multiplying (4.13) by φ and rearranging yields equation (4.14). ■

Remark 4.5

Theorem 4.2 does not assert that the general warped-product construction is new; warped-product gradient almost Ricci solitons have already been investigated in broader dimensions and settings [4]. Its role here is to provide an explicit, complete and easily verifiable two-dimensional model, suitable for geometric analysis and visualisation.

5. Killing Fields and Radial Soliton Potentials

Rotational surfaces naturally admit angular symmetries.

Proposition 5.1. Rotational Killing field

For every metric

$g = dr^2 + \varphi(r)^2 d\theta^2$,

the vector field

$Y = \partial\theta$

is Killing.

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Proof

The coefficients of g are independent of θ. Therefore, translation in the θ-direction preserves the metric. Equivalently,

$\mathscr{L}_{\partial\theta}\, g = 0.$

Hence $\partial\theta$ is Killing. ■

The angular field $\partial\theta$ generates the rotational symmetry of the surface. By contrast, a radial vector field is generally not Killing.

Proposition 5.2. Radial Killing fields

Let

$V = q(r)\partial r$

be a radial vector field on M. Then

$\mathscr{L}_V\, g = 2q' dr^2 + 2q\varphi\varphi' d\theta^2$. (5.1)

Consequently, a non-zero radial vector field is Killing if and only if q is constant and φ is constant on the relevant connected region. Therefore, a non-zero radial Killing field occurs only on a cylindrical product region.

Proof

Using the coordinate expression for the Lie derivative,

$(\mathscr{L}_V\, g)(\partial r,\partial r) = 2g(\nabla_{\partial r} V,\partial r).$

Since

$\nabla_{\partial r} V = q'\partial r,$

we obtain

$(\mathscr{L}_V g)(\partial r,\partial r) = 2q'.$

Similarly,

$\nabla_{\partial\theta} V = q\nabla_{\partial\theta} \partial r = q(\varphi'/\varphi)\partial\theta,$

so

$(\mathscr{L}_V g)(\partial\theta,\partial\theta)$
$= 2g(q(\varphi'/\varphi)\partial\theta,\partial\theta)$
$= 2q\varphi\varphi'.$

Thus equation (5.1) holds.

If V is Killing, then $q' = 0$ and $q\varphi\varphi' = 0$. Since $\varphi > 0$, a non-zero q forces $\varphi' = 0$. Hence φ is constant and the metric is locally cylindrical. The converse is immediate. ■

Corollary 5.3. When is the soliton potential field Killing?

For the radial gradient almost Ricci soliton of Theorem 4.2 with $\mu \neq 0$, the potential vector field

$\nabla f = \mu\varphi\partial r$

is Killing if and only if $\varphi' = 0$. Hence, except on a cylindrical region, the soliton potential is conformal but not Killing.

Proof

By Corollary 4.3,

$\mathscr{L}_{\nabla f}\, g = 2\mu\varphi' g.$

For $\mu \neq 0$, this vanishes exactly when $\varphi' = 0$. ■

This distinction is geometrically significant: rotational symmetry is generated by the angular Killing field $\partial\theta$, whereas the radial soliton potential typically produces a conformal deformation of the metric.

6. Compact Sphere-Type Rotational Surfaces

We now examine a smooth rotational metric on a two-sphere. Let

$M = [0,L] \times S^1$

with metric

$g = dr^2 + \varphi(r)^2 d\theta^2$,

where the endpoints correspond to poles. Smoothness at the poles requires

$\varphi(0) = 0,\ \varphi(L) = 0,$ (6.1)

$\varphi'(0) = 1,\ \varphi'(L) = -1,$ (6.2)

and

$\varphi(r) > 0$ for $0 < r < L$. (6.3)

The area form is

$dA = \varphi(r)\, dr\, d\theta$. (6.4)

Theorem 6.1. Total curvature and mean soliton function

Let M be a smooth sphere-type rotational surface satisfying conditions (6.1)–(6.3). Suppose that it admits the radial gradient almost Ricci soliton structure

$f' = \mu\varphi$,

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$\lambda = -\varphi''/\varphi + \mu\varphi'$.

Then

$\int_M K\, dA = 4\pi$, (6.5)

and

$\int_M \lambda\, dA = 4\pi$. (6.6)

Consequently,

$\text{average}(\lambda) = 4\pi / \text{Area}(M)$. (6.7)

Proof

Using $K = -\varphi''/\varphi$ and $dA = \varphi\, dr\, d\theta$,

$$\int_M K\, dA$$
$$= \int_0^{2\pi} \int_0^L (-\varphi''/\varphi)\varphi\, dr\, d\theta$$
$$= -2\pi \int_0^L \varphi''\, dr$$
$$= -2\pi[\varphi'(L) - \varphi'(0)].$$

By condition (6.2),

$\varphi'(L) - \varphi'(0) = -1 - 1 = -2$.

Therefore,

$\int_M K\, dA = 4\pi$.

This is also the Gauss–Bonnet identity for a two-sphere.

Next,

$\lambda = K + \mu\varphi'$.

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Therefore,

$\int_M \lambda \, dA$
$= \int_M K \, dA + \mu \int_M \varphi' \, dA$
$= 4\pi + 2\pi\mu \int_0^L \varphi\varphi' \, dr.$

Since

$\int_0^L \varphi\varphi' \, dr = \tfrac{1}{2}[\varphi(r)^2]_0^L,$

conditions $\varphi(0) = \varphi(L) = 0$ give

$\int_0^L \varphi\varphi' \, dr = 0.$

Hence,

$\int_M \lambda \, dA = 4\pi.$

Dividing by Area(M) yields equation (6.7). ■

Proposition 6.2. Mean-zero deviation from curvature

Under the hypotheses of Theorem 6.1,

$\lambda - K = \mu\varphi'$ (6.8)

and

$\int_M (\lambda - K) dA = 0.$ (6.9)

Moreover,

$\int_M (\lambda - K)^2 \, dA$
$= 2\pi\mu^2 \int_0^L \varphi(r)\varphi'(r)^2 \, dr.$ (6.10)

Proof

Equation (6.8) follows from Theorem 4.2. Integrating,

$\int_M (\lambda - K)dA$
$= 2\pi\mu\int_0^L \varphi\varphi' \, dr$
$= \pi\mu[\varphi^2]_0^L$
$= 0.$

Similarly,

$\int_M (\lambda - K)^2 dA$
$= \int_0^{2\pi} \int_0^L \mu^2\varphi'^2\varphi \, dr \, d\theta$
$= 2\pi\mu^2\int_0^L \varphi\varphi'^2 \, dr.$

This establishes the result. ■

Proposition 6.3. Pole-to-pole variation of the potential

For a sphere-type rotational almost soliton,

$f(L) - f(0) = \mu \, \mathrm{Area}(M)/(2\pi)$. (6.11)

Proof

Since $f' = \mu\varphi$,

$f(L) - f(0)$
$= \mu\int_0^L \varphi(r) \, dr.$

But

$\mathrm{Area}(M) = 2\pi\int_0^L \varphi(r) \, dr.$

Therefore,

$f(L) - f(0) = \mu \,\mathrm{Area}(M)/(2\pi)$. ■

Equation (6.11) gives a direct geometric meaning to the constant μ: it measures the potential difference between the two poles after normalisation by the surface area.

7. Explicit Examples

7.1 Euclidean plane

Consider the polar form of the Euclidean metric:

$g = dr^2 + r^2 d\theta^2$,

where, $\varphi(r) = r$, $r > 0$.

Then $\varphi' = 1$, $\varphi'' = 0$,

and hence

$K = -\varphi''/\varphi = 0$.

By Theorem 4.2,

$f' = \mu r$,

so

$f(r) = \frac{1}{2}\mu r^2 + b$.

Furthermore,

$\lambda = K + \mu\varphi' = \mu$.

Thus,

$Ric + Hess\ f = \mu g$.

Therefore, the Euclidean plane admits a radial gradient Ricci soliton with quadratic potential function

$f(r) = \frac{1}{2}\mu r^2 + b$.

In Cartesian coordinates, since $r^2 = x^2 + y^2$,

$f(x,y) = \frac{1}{2}\mu(x^2 + y^2) + b$,

and

$Hess\ f = \mu g$.

This is an ordinary Ricci soliton because λ is constant.

7.2 Circular cylinder

Let

$\varphi(r) = R$,

where $R > 0$ is constant. Then

$g = dr^2 + R^2 d\theta^2$

is the standard product metric on a cylinder.

Since

$\varphi' = 0$, $\varphi'' = 0$,

we obtain

$K = 0.$

Theorem 4.2 gives

$f' = \mu R,$

so

$f(r) = \mu R r + b,$

and

$\lambda = 0.$

Moreover,

$\mathrm{Hess}\, f = 0.$

Thus,

$\mathrm{Ric} + \mathrm{Hess}\, f = 0.$

The potential vector field

$\nabla f = \mu R \partial r$

is parallel and Killing. This agrees with Proposition 5.2: the cylinder is precisely the non-trivial rotational geometry admitting a radial Killing field.

7.3 Standard sphere

Let $S^2(R)$ denote the round sphere of radius R. In geodesic polar coordinates,

$g = dr^2 + R^2 \sin^2(r/R)d\theta^2,$

where

$0 < r < \pi R.$

Thus,

$\varphi(r) = R \sin(r/R).$

Then

$\varphi'(r) = \cos(r/R),$

$\varphi''(r) = -(1/R)\sin(r/R),$

and therefore

$K = -\varphi''/\varphi = 1/R^2.$

For any constant μ,

$f'(r) = \mu R \sin(r/R).$

Integrating,

$f(r) = -\mu R^2 \cos(r/R) + b.$

The soliton function is

$\lambda(r) = 1/R^2 + \mu \cos(r/R).$ (7.1)

If $\mu = 0$, then f is constant and $\lambda = 1/R^2$; this is the usual Einstein structure of the round sphere.

If $\mu \neq 0$, then λ varies smoothly from

$1/R^2 + \mu$

at the north pole to

$1/R^2 - \mu$

at the south pole. Hence, the sphere carries a non-trivial gradient almost Ricci soliton with conformal potential vector field.

Indeed,

$\nabla f = \mu R \sin(r/R)\partial r$

and

$\mathscr{L}_{\nabla f}\, g = 2\mu \cos(r/R)g.$

The integral identity of Theorem 6.1 is verified directly:

$\mathrm{Area}(S^2(R)) = 4\pi R^2,$

and

$\mathrm{average}(\lambda) = 1/R^2 = 4\pi/(4\pi R^2).$

7.4 Hyperbolic plane

Let $H^2(R)$ denote the hyperbolic plane of curvature $-1/R^2$, expressed in rotational coordinates:

$g = dr^2 + R^2 \sinh^2(r/R)d\theta^2.$

Then

$\varphi(r) = R \sinh(r/R),$

$\varphi'(r) = \cosh(r/R),$

$\varphi''(r) = (1/R)\sinh(r/R).$

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Therefore,

$K = -1/R^2$.

The radial potential is

$f'(r) = \mu R \sinh(r/R)$,

hence

$f(r) = \mu R^2 \cosh(r/R) + b$.

The soliton function is

$\lambda(r) = -1/R^2 + \mu \cosh(r/R)$. (7.2)

For $\mu = 0$, the structure reduces to the Einstein hyperbolic metric. For $\mu \neq 0$, λ is non-constant and increases radially.

7.5 A catenoid-type warped surface

Consider

$\varphi(r) = \sqrt{(a^2 + r^2)}$,

where $a > 0$ and $r \in \mathbb{R}$. Then

$\varphi'(r) = r/\sqrt{(a^2 + r^2)}$,

and

$\varphi''(r) = a^2/(a^2 + r^2)^{3/2}$.

Therefore,

$K(r) = -a^2/(a^2 + r^2)^2$. (7.3)

The potential function is obtained from

$f'(r) = \mu\sqrt{(a^2 + r^2)}$.

Using the elementary integral,

$\int \sqrt{(a^2 + r^2)}\, dr$
$= \frac{1}{2}[r\sqrt{(a^2 + r^2)} + a^2 \ln(r + \sqrt{(a^2 + r^2)})]$,

we obtain

$f(r)$
$= (\mu/2)[r\sqrt{(a^2 + r^2)}$

- $a^2 \ln(r + \sqrt{(a^2 + r^2)})] + b$. (7.4)

The soliton function is

$\lambda(r)$
$= -a^2/(a^2 + r^2)^2$

- $\mu r/\sqrt{(a^2 + r^2)}$. (7.5)

This example exhibits a non-compact warped surface with negative curvature and an asymmetric soliton function when $\mu \neq 0$.

8. Surfaces of Revolution and Geometric Visualisation

A rotational warped metric may often be realised as a surface of revolution in Euclidean three-space. Suppose that

$|\varphi'(r)| \leq 1$.

Define

$z'(r) = \sqrt{(1 - \varphi'(r)^2)}$.

Then the map

$X(r,\theta)$
$= (\varphi(r)\cos\theta, \varphi(r)\sin\theta, z(r))$ (8.1)

has first fundamental form

$dX \cdot dX$
$= [\varphi'(r)^2 + z'(r)^2]dr^2 + \varphi(r)^2 d\theta^2$
$= dr^2 + \varphi(r)^2 d\theta^2$.

Thus, equation (8.1) provides an isometric realisation of the warped metric as a Euclidean surface of revolution whenever the stated condition holds.

This representation is appropriate for the sphere and catenoid-type examples. The complete hyperbolic-plane model does not admit such a global Euclidean surface-of-revolution realisation because its warping derivative exceeds one for positive r.

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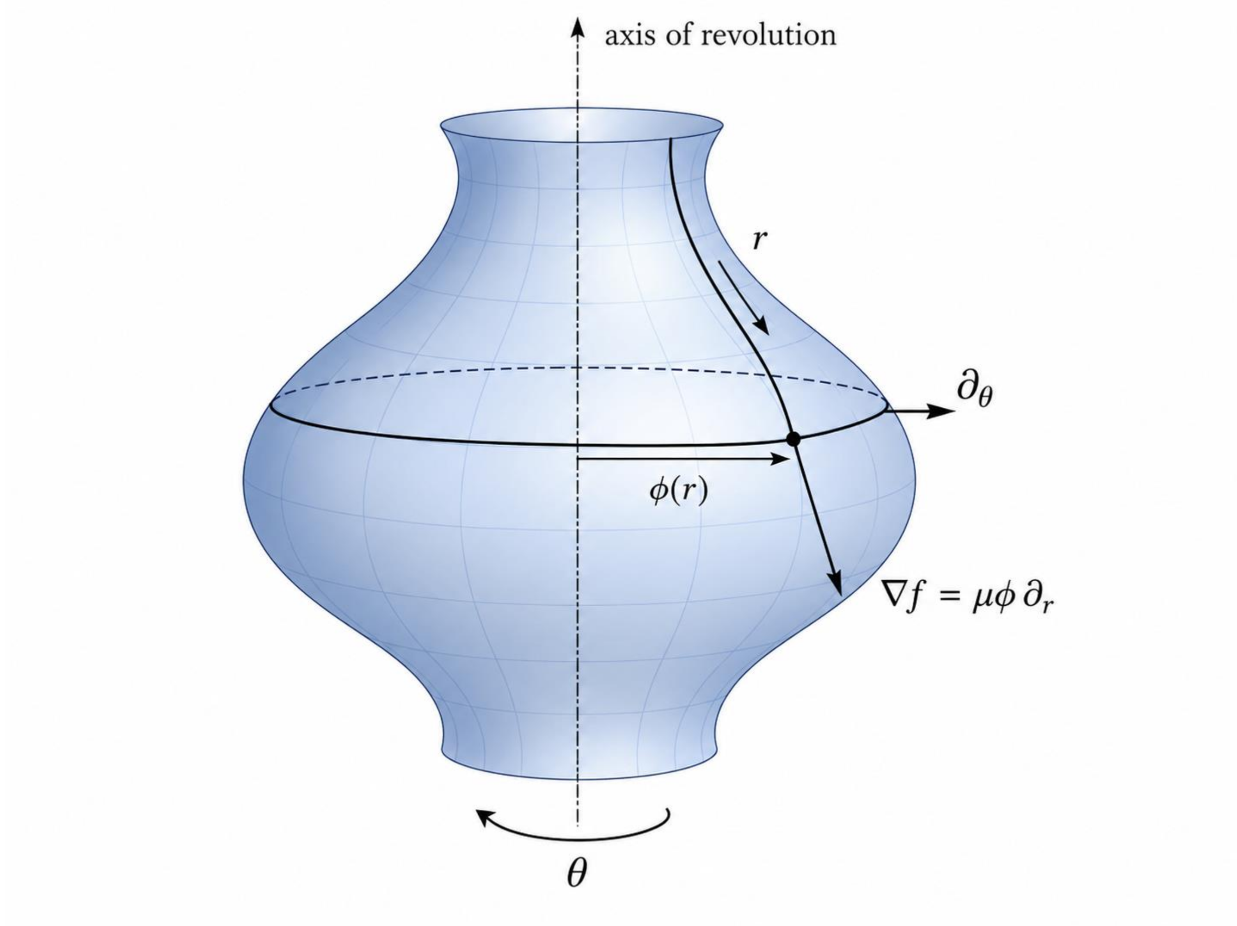

Figure 1. Rotational warped surface and coordinate vector fields

A smooth surface of revolution with meridian coordinate r and angular coordinate θ. Indicate:

- the angular Killing field $\partial\theta$ tangent to circles of revolution;
- the radial potential field $\nabla f = \mu\varphi\partial r$ tangent to meridian curves;
- the radius from the axis equal to $\varphi(r)$.

Killing vector field $\partial\theta$, while the radial gradient almost Ricci soliton potential is $\nabla f = \mu\varphi\partial r$.

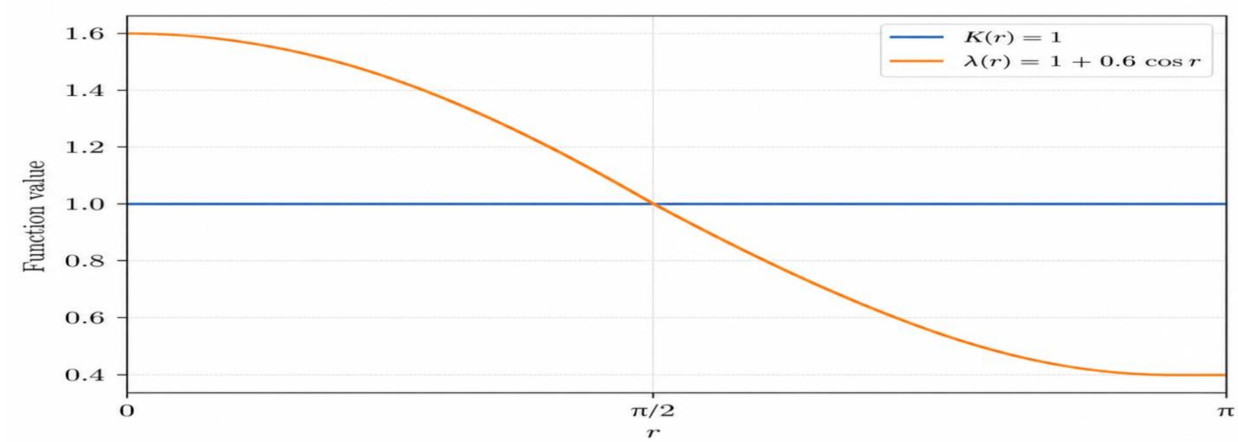

Figure 2. Gaussian curvature and soliton function for the almost Ricci soliton on the unit sphere. Although the curvature is constant, the almost-soliton function varies with latitude when $\mu \neq 0$.

Figure 2. Soliton function on the unit sphere

Gaussian curvature and soliton function for the almost Ricci soliton on the unit sphere. Although the curvature is constant, the almost-soliton function varies with latitude when $\mu \neq 0$.

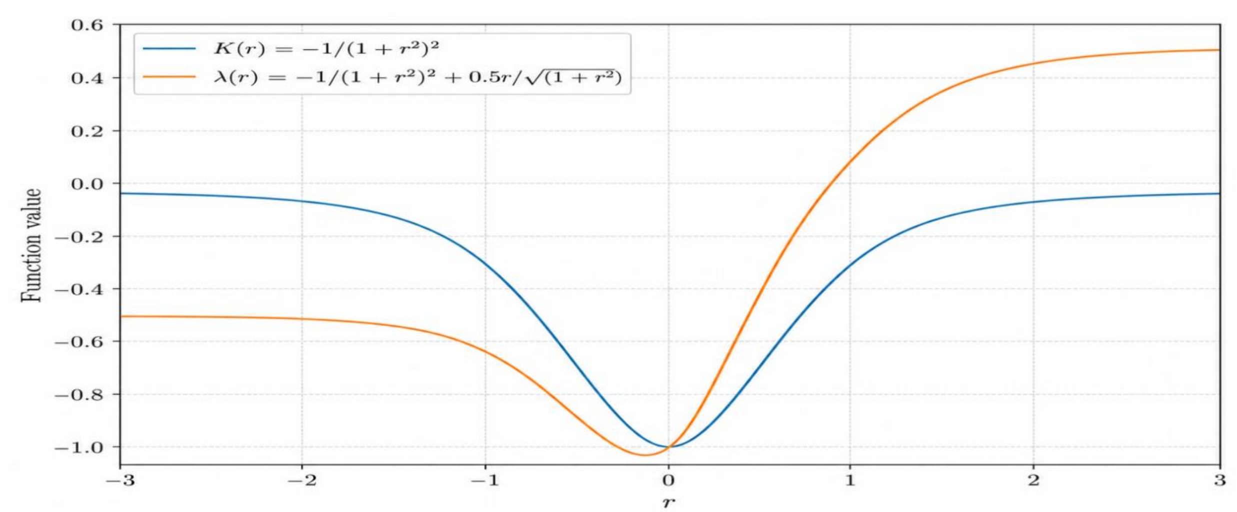

Figure 3. Comparison between curvature and soliton function on a catenoid-type warped surface. The term $\mu\phi'$ changes the symmetry and asymptotic behaviour of λ.

Figure 3. Curvature and soliton function for the catenoid-type example

Comparison between curvature and soliton function on a catenoid-type warped surface. The term $\mu\varphi'$ changes the symmetry and asymptotic behaviour of λ.

9. Discussion

The central result of this paper is the reduction of the gradient almost Ricci soliton equation on a rotational warped surface to the explicit system

$f' = \mu\varphi,$

$\lambda = -\varphi''/\varphi + \mu\varphi'.$

This reduction makes several geometric relationships transparent.

First, curvature and soliton behaviour separate naturally. The term

$-\varphi''/\varphi$

is the intrinsic Gaussian curvature of the surface, whereas

$\mu\varphi'$

records the contribution of the radial conformal potential field. Thus, λ is not an arbitrary auxiliary function: it is determined by the balance between intrinsic curvature and radial geometric deformation.

Second, the potential vector field is necessarily conformal. This property connects the present model with rigidity questions studied for almost Ricci solitons whose potential fields possess additional symmetry or conformality assumptions. In the spherical example, the non-trivial potential function is proportional to the height function, a classical source of conformal gradient fields on round spheres.

Third, Killing symmetry and soliton symmetry play different roles. The field $\partial\theta$ is Killing on every rotational warped surface because it preserves the angular symmetry. However, the radial field ∇f is generally not Killing; it is conformal and becomes Killing only in cylindrical regions where φ is constant.

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Fourth, the compact integral identity

$$\int_M \lambda \, dA = 4\pi$$

shows that, on sphere-type rotational surfaces, the average soliton function is topologically constrained. The variable contribution $\mu\varphi'$ integrates to zero. Consequently, variation in λ redistributes the local soliton behaviour without changing its total integral.

The present work should be positioned carefully in relation to existing literature. General constructions of gradient Ricci almost solitons on warped products were already developed by Feitosa et al. [4]. Compact rigidity questions for gradient almost Ricci solitons have also been treated by Barros, Batista and Ribeiro Jr. [5], and recent work summarised by Blaga includes further results on potential fields and geometric structures [1,2]. Therefore, before journal submission, an exhaustive novelty comparison must determine whether the explicit classification and integral identities developed here are already contained, implicitly or explicitly, in previous specialised treatments.

Nevertheless, the manuscript contributes a coherent two-dimensional analysis with complete computations, explicit examples and direct geometric interpretation. It is substantially stronger than a descriptive review because it formulates a specific differential-geometric problem and proves all central statements used in the analysis.

10. Conclusion

This paper has examined gradient almost Ricci soliton structures on rotational warped surfaces with metric

$$g = dr^2 + \varphi(r)^2 d\theta^2.$$

For radial potential functions, the soliton equation is completely characterised by

$$f' = \mu\varphi$$

and

$\lambda = -\varphi''/\varphi + \mu\varphi'$.

The associated potential vector field is conformal, whereas the natural angular field is Killing. On smooth rotational metrics of sphere type, the soliton function satisfies the global identity

$\int_M \lambda \, dA = 4\pi$,

and its average is therefore determined by the surface area.

The explicit calculations on the Euclidean plane, cylinder, sphere, hyperbolic plane and catenoid-type surface demonstrate how curvature and soliton deformation interact in concrete geometries. These results provide a focused differential-geometric study suitable for further development into a journal manuscript after full novelty verification, journal-specific formatting, authorship completion and professional mathematical review.

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