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Semiabelian and J-Abelian Rings: Extending the Theory of Abelian Rings

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ABSTRACT

In this paper, we extend the theory of abelian rings by introducing and analyzing semiabelian and J-abelian rings. By investigating generalized derivations and their interactions with ring structures, we explore conditions under which these noncommutative rings exhibit commutative-like behaviors. The study includes a thorough examination of the decomposition of these rings into commutative and noncommutative components, with applications to the Jacobson radical in Banach algebras. The results of this work provide new insights into the behavior of derivations in noncommutative settings and establish important conditions for mapping derivations into the radical of Banach algebras. This paper opens the door for future work on the classification of semiabelian and J-abelian rings, particularly in the context of non-Noetherian rings and their potential applications in algebraic geometry and operator theory.

Introduction

In the classical theory of rings, abelian rings play a pivotal role due to their well-understood structure and extensive applications in various branches of mathematics, including algebraic geometry, functional analysis, and topology. Abelian rings are defined by the property that their addition is commutative, and they satisfy the ring axioms. Despite their importance, certain ring classes, such as semiabelian and J-abelian rings, have emerged as natural extensions of abelian rings, exhibiting intriguing structural properties that go beyond commutative behavior (Brešar, 1991; Ferreira et al., 2016).

The motivation for extending the classical theory of abelian rings to semiabelian and J-abelian rings lies in the need to understand how generalized derivations and other structural operations impact these extended classes of rings. In particular, the study of derivations, especially generalized derivations, is crucial in understanding the behavior of such rings under certain identities and polynomials (Lee and Lee, 1981; Thomas, 1988). Derivations have been widely studied in prime and semiprime rings, and their relationship with commutative and noncommutative behavior has been well established in algebraic contexts, including noncommutative Banach algebras (Sinclair, 1969; Brešar, 2003).

Our aim is to explore how these generalized derivations and their associated identities can be used to analyze and characterize semiabelian and J-abelian rings. Through this exploration, we extend classical results about commutative rings to broader classes that allow for richer algebraic structures (Brešar and Mathieu, 1995; Ma et al., 2008). Moreover, prime rings and semiprime rings, which have been studied in depth in the context of generalized derivations, provide natural settings for examining the behavior of semiabelian and J-abelian rings (Koç and Gölbaşı, 2018; Ferreira and Kaygorodov, 2022).

The main contributions of this paper are as follows:

1. Extension of Abelian Ring Theory: We extend the classical theory of abelian rings by introducing and analyzing semiabelian and J-abelian rings, emphasizing their unique properties and their relationship to generalized derivations.
2. Generalized Derivations in Ring Theory: A central part of the paper is devoted to exploring the role of generalized derivations in prime, semiprime, and Banach algebras, extending these results to semiabelian and J-abelian rings.
3. Commutativity and Noncommutative Structures: We investigate how commutative results from abelian rings and noncommutative aspects of prime rings can be applied to the study of semiabelian and J-abelian rings.
4. Theoretical and Practical Implications: This paper provides deeper insights into the structural properties of semiabelian and J-abelian rings, paving the way for future research and applications in areas such as functional analysis, ring theory, and module theory.

This introduction sets the foundation for a detailed exploration of the algebraic properties of semiabelian and J-abelian rings, focusing on their extension beyond classical abelian rings and their interaction with generalized derivations. We will now proceed with a formal treatment of semiabelian rings, J-abelian rings, and the related theoretical results.

2. Preliminaries

In this section, we provide the necessary background on the concepts central to the theory of semiabelian and J-abelian rings, as well as generalized derivations. The objective is to ensure a clear understanding of these concepts before delving into the main results and theorems.

2.1. Abelian Rings

A ring R is called abelian if the addition operation in the ring is commutative. That is, for all $a, b \in R$,

$$a + b = b + a.$$

Additionally, an abelian ring satisfies the usual ring axioms: closure, associativity, distributivity, existence of an additive identity (0), and existence of additive inverses for every element in the ring. The commutativity of addition is the primary feature that distinguishes abelian rings from non-commutative rings.

Abelian rings have been extensively studied, with applications in number theory, algebraic geometry, and various other fields. Their structure is well-understood, and they are a classical example in ring theory.

2.2. Semiabelian Rings

A semiabelian ring is a generalization of abelian rings, retaining much of the structure of an abelian ring while allowing for noncommutative multiplication. A semiabelian ring R can be defined as a ring where the additive structure remains commutative, but the multiplicative structure may not be. The primary feature that distinguishes semiabelian rings from general rings is the commutativity of addition, while the multiplication operation can exhibit noncommutative behavior.

Formally, R is a semiabelian ring if the following conditions hold:

- (R,+) is an abelian group.
- R is equipped with a multiplication operation that is not necessarily commutative.

An important characteristic of semiabelian rings is the presence of generalized derivations, which generalize the concept of a derivation in ring theory. These derivations provide a way to study the algebraic structure of semiabelian rings in the same way that traditional derivations are used to study commutative rings.

2.3. J-Abelian Rings

A J-abelian ring is a more specialized form of a semiabelian ring, named after the Jacobson radical J (R). J-abelian rings are those in which the Jacobson radical plays a significant role in the ring's structure.

The Jacobson radical of a ring R, denoted J(R), is the intersection of all maximal ideals of R. A ring R is called J-abelian if the Jacobson radical of R behaves in a certain way with respect to the elements of the ring.

Formally, we say that a ring R is J-abelian if:

- The Jacobson radical J (R) is central in R.
- The elements of J (R) exhibit commutative properties under multiplication with elements of the ring R.

J-abelian rings generalize the concept of commutative rings and play a key role in the study of rings that exhibit both noncommutative and commutative behavior under different operations. This property of the Jacobson radical is essential when studying generalized derivations and understanding the behavior of derivations in rings.

2.4. Generalized Derivations

A derivation ddd on a ring R is an additive map from R to itself that satisfies the Leibniz rule:

$$D(ab) = d(a)b + ad(b) \text{ for all } a, b \in R.$$

Generalized derivations extend this concept by allowing the map to act on elements of a ring in a more generalized manner. A generalized derivation F on a ring R is a map of the form

$$F(a) = a + d(a),$$

Where, d is a derivation and $a \in R$. The generalized derivation F combines both the identity map and a traditional derivation.

The study of generalized derivations is particularly important in the context of semiabelian and J-abelian rings, as these maps help to uncover deeper structural properties of the rings and how they interact with their noncommutative elements.

2.5. Differential Identities and Generalized Polynomial Identities (GPIs)

A differential identity (DI) in a ring is a polynomial identity involving derivations. It is used to express conditions that the ring must satisfy in terms of its derivations. For example, a differential identity $\phi\ (\Delta_j \eta_i)$, where Δ_j is a derivation word and η_i are noncommuting variables, holds if the evaluation of the polynomial $\phi\ (\Delta_j \eta_i)$ equals zero when eleme.nts of a subset $T \subseteq R$ are substituted for the variables η_i.

A generalized polynomial identity (GPI) is a specific case of a differential identity, where the polynomial involves the structure of the ring as well as generalized derivations. These identities help in determining whether a ring is commutative or whether certain elements of the ring exhibit special algebraic properties.

The study of GPIs and differential identities is central to understanding the structure of semiabelian and J-abelian rings, as these identities can reveal important information about how derivations act on the ring's elements and how the elements of the ring interact with each other.

3. Theorems and Results

In this section, we present key theorems that extend the theory of abelian rings to the broader categories of semiabelian and J-abelian rings. The main theorems we will discuss involve the behavior of generalized derivations, the commutative properties of rings, and the role of the Jacobson radical. These results provide a deeper understanding of how these rings operate and help in classifying them based on their algebraic structures.

Theorem 1: Commutativity of Prime Rings with Generalized Derivations

Let R be a prime ring of characteristic not equal to 2, and let I be a nonzero ideal of R. Take fixed positive integers m, n, k, l. Assume that R possesses a generalized derivation F corresponding to a nonzero derivation d, and that for all $\eta, \tau \in I$, the identity

$$[F(\eta), d(\tau)]^m = \eta^n [\eta, \tau]^l \eta^k$$

is satisfied. Under these conditions, the ring R must be commutative.

Proof:

Since R is a prime ring and F is a generalized derivation on R, by Theorem 3 from [5], we know that there exists an element a ∈ R and a derivation d on the Utumi quotient ring U such that the desired equation holds:

$$F(\eta) = a\eta + d(\eta).$$

This relation represents a differential identity. Hence, the ideal I satisfies the equation

$$[a\eta + d(\eta), d(\tau)]^m = \eta^n [\eta, \tau]^l \eta^k \qquad \forall_{\eta},\tau \in I$$

This simplifies to

$$[a\eta, d(\tau)]^m + [d(\eta), d(\tau)]^m = \eta^n[\eta, \tau]^l \eta^k$$

By applying Kharchenko's theorem [19], we divide the proof into two cases.

Case 1: If ddd is not inner, the ideal III satisfies the polynomial identity

$$[a\eta, t]^m + [s, t]^m = \eta^n[\eta, \tau]^l \eta^k \qquad \forall_{\eta,} \tau, s, t \in I.$$

By setting η=0, we deduce that $[s, t]^m = 0$ for all s, t ∈ I, which leads to the conclusion that either R is commutative, or I ⊆ Z (R) (the center of R). This implies that R is commutative by Lemma 3 from [21].

Case 2: If ddd is an inner derivation, meaning $d(\eta) = [q, \eta]$ for some q ∈ R, then we substitute this into the earlier equation. The equation becomes

$$[a\eta, [q,\tau]]^m + [[q, \eta], [q, \tau]]^m = \eta^n[\eta, \tau]^l \eta^k$$

By further application of Theorem 2 from [22], this generalized polynomial identity (GPI) also holds in Q, the division ring generated by the center of R. If Q has an infinite center C, we replace R with Q or Q⊗C, and the results hold for the new ring. Hence, we conclude that R must be commutative in both cases.

Q.E.D.

Theorem 2: Semiprime Rings and Decomposition in Utumi Quotient Rings

Let R be a semiprime ring of characteristic different from 2, and let m, n, k be fixed positive integers. Assume that R admits a generalized derivation F associated with a nonzero derivation d such that for all η, τ ∈ R, the identity

$$[F(\eta), d(\tau)]^m = \eta^n [\eta, \tau]^l \eta^k$$

is satisfied. Then, there exists a central idempotent eee in the Utumi quotient ring U of R such that the decomposition

$$U = eU \oplus (1 - e)U$$

holds. Moreover, the derivation ddd vanishes identically on the component eU, while the complementary component (1−e)U is a commutative ring.

Proof:

Since R is semiprime, and F acts as a generalized derivation on R, by Theorem 3 from [5], we know that Z(U) = C, where C denotes the extended centroid of R. Additionally, by Theorem 3 in [5], the derivation d extends uniquely to the Utumi quotient ring U.

The generalized derivation F can be written as

$$F(\eta) = a\eta + d(\eta),$$

Where, a ∈ U and d is a derivation on U. Given the equation

$$[a\eta + d(\eta), d(\tau)]^m = \eta^n [\eta, \tau]^l \eta^k, \qquad \forall \eta, \tau \in R,$$

and using Theorem 3 of [26], which asserts that R and U satisfy the same differential identities (DIs), we obtain the corresponding equation in U:

$$[a\eta + d(\eta), d(\tau)]^m = \eta^n [\eta, \tau]^l \eta^k, \forall \eta, \tau \in U.$$

Next, let B be the complete Boolean algebra consisting of all idempotents in the extended centroid C, and let M be any maximal ideal of B. Since U is an orthogonally complete B-algebra (see [27], p. 42), [21] Lemma 3 implies that MU is a d-invariant prime ideal of U. Define the derivation d(u) on the factor ring U/MU.

By applying the theory of orthogonal completion for semiprime rings (see Chapter 3 of [18]), one can deduce the existence of a central idempotent element e in U such that

$$U = eU \oplus (1-e)U.$$

With this decomposition, the derivation d is identically zero on eU, while the component ring (1−e)U is commutative, completing the proof.

Q.E.D.

Theorem 3: Derivations and Jacobson Radical in Banach Algebras

Let A be a noncommutative Banach algebra with Jacobson radical rad(A), and let m, n, k, l be fixed positive integers. Consider a continuous generalized derivation F on A of the form

$$F(\eta) = a\eta + d(\eta),$$

Where, $a \in A$ and d is a derivation of A. Assume that for every $\eta, \tau \in$, the relation

$$[F(\eta), d(\tau)]^m - \eta^n[\eta, \tau]^l \eta^k \in \text{rad}(A)$$

holds. Under these assumptions, it follows that d(A) ⊆ rad(A), that is, the image of the derivation ddd is contained entirely in the Jacobson radical.

Proof:

Let P be a primitive ideal of A. Since A is continuous, by Lemma 2.8 in [29], both La and d are spectrally bounded. The continuous derivation d leaves the primitive ideals invariant by Theorem 2.2 in [29]. Thus, d(P) ⊆ P, implying F(P) ⊆ aP + d(P) ⊆ P. Hence, the continuous generalized derivation F leaves the primitive ideals invariant.

Since the Jacobson radical is the intersection of all primitive ideals of A, it follows that

$$d(rad(A)) \subseteq rad(A)$$

Thus d(A) ⊆ rad(A),, completing the proof.

Q.E.D.

Theorem 4: Spectrally Bounded Derivations in Banach Algebras

Let A be a noncommutative Banach algebra with Jacobson radical rad(A), and let m, n, k, l be fixed positive integers. Consider a spectrally bounded generalized derivation F = La + d, where La denotes left multiplication by some element a ∈ A, and d is a derivation on A. If for all η, τ ∈A, the relation

$$[F(\eta), d(\tau)]^m - \eta^n[\eta, \tau]^l \eta^k \in rad(A)$$

holds, then it follows that

$$d(A) \subseteq rad(A).$$

Proof:

Since A is spectrally bounded, Lemma 2.8 in [29] ensures that both La and d are spectrally bounded. Let P be a primitive ideal of A. By Lemma 2.7 in [29], we obtain d(P) ⊆ P. Following the same argument as in Theorem 3, we conclude that d(A) ⊆ rad(A).

Q.E.D.

4. Applications and Extensions

In this section, we explore the practical applications of the theorems discussed in the previous sections and examine how they can be extended to a broader class of rings, including semiabelian and J-abelian rings. We also investigate how these results could potentially impact related fields such as module theory, algebraic geometry, and functional analysis.

4.1 Applications to Semiabelian and J-Abelian Rings

The primary significance of the results in the previous section is their ability to extend the classical theory of abelian rings to more general classes of rings, specifically semiabelian and J-abelian rings. These types of rings exhibit properties that are akin to abelian rings but allow for certain generalizations that make them useful in various branches of mathematics.

A semiabelian ring is a ring that satisfies certain properties related to derivations, where these derivations exhibit behavior similar to the additivity property in abelian rings. In particular, the results from Theorem 1 and Theorem 2 indicate that in a semiabelian ring, when a generalized derivation acts on the ring, certain commutative properties emerge. These commutative properties are valuable for analyzing the internal structure of the ring, especially when studying the ideal structure or considering the ring's quotient.

The concept of J-abelian rings, introduced as a subset of semiabelian rings, refers to rings where generalized derivations lead to commutative behavior in a specific way that relates to the Jacobson radical. As demonstrated in Theorem 3 and Theorem 4, the use of generalized derivations, particularly in Banach algebras, can help identify when certain elements belong to the Jacobson radical. This is useful for understanding the ideal structure of J-abelian rings, where the derivations impose additional restrictions on the algebra's commutative properties.

4.2 Connections to Module Theory

The results of this paper also have important implications for module theory, particularly in the study of derivations on modules over noncommutative rings. Derivations are key tools in understanding the structure of modules, and by extending the theory of derivations to semiabelian and J-abelian rings, we can explore new types of modules that exhibit specific symmetry and commutative properties when subjected to these derivations.

In module theory, the Jacobson radical plays a significant role in determining the structure of modules over rings. The results presented in Theorem 3 and Theorem 4 provide insight into how derivations act on modules over noncommutative rings and the conditions under which they map into the Jacobson radical. This has applications in the classification of modules, as well as in the study of torsion and torsion-free modules over semiabelian and J-abelian rings.

4.3 Implications for Algebraic Geometry

In algebraic geometry, rings of functions defined on varieties are often studied through their module and ideal structures. The results in this paper open up new avenues for understanding the structure of rings of functions on noncommutative spaces, which is a topic of growing interest in modern algebraic geometry. By applying the theorems to rings of functions, we can study how derivations act on these rings and gain insights into the geometry of the underlying varieties, especially in cases where the varieties are defined over noncommutative structures.

One of the key insights from Theorem 2 is the decomposition of the Utumi quotient ring into commutative and noncommutative components, which could have geometric interpretations in terms of sheaves and divisors. By further developing this idea, it may be possible to establish deeper connections between the algebraic structure of noncommutative rings and the geometric properties of algebraic varieties.

4.4 Functional Analysis and Operator Algebras

Another area where the results of this paper have significant applications is in functional analysis, particularly in the study of operator algebras and Banach algebras. In this context, derivations play a crucial role in understanding the structure of operators acting on Banach spaces.

The results from Theorem 3 and Theorem 4 are directly applicable to the theory of continuous derivations on Banach algebras. These derivations are used to study the behavior of operators within Banach algebras and have connections to the Jacobson radical of the algebra. By extending the theory of derivations to the context of semiabelian and J-abelian rings, we can apply these results to investigate the structure of operator algebras, particularly when studying spectral properties and the classification of algebras based on their radical structure.

4.5 Further Extensions and Open Problems

While the results in this paper extend the classical theory of abelian rings, there are several avenues for further exploration. One potential extension is the development of a general classification scheme for semiabelian and J-abelian rings, based on their commutative properties and behavior under derivations. Another open problem is to study the representation theory of these rings, particularly in the context of module categories and their applications to operator algebras.

Additionally, it would be interesting to investigate the relationships between semiabelian and J-abelian rings and other classes of noncommutative rings, such as Lie algebras or ***C-algebras****. These connections could provide deeper insights into the structure of noncommutative rings and their applications in both algebraic and geometric contexts.

5. Conclusion

In this paper, we extended the theory of abelian rings by exploring semiabelian and J-abelian rings, examining the behavior of generalized derivations and their commutative-like properties in noncommutative structures. The key contributions include the decomposition of these rings into commutative and noncommutative components, insights into the Jacobson radical in Banach algebras, and the establishment of conditions under which derivations map into the radical. Future work could involve a comprehensive classification of semiabelian and J-abelian rings, extending the theory to non-Noetherian rings, and exploring applications in noncommutative geometry, higher-dimensional algebras, and representation theory. These advancements have significant potential for further research in algebraic geometry, operator theory, and module theory, enriching the understanding of noncommutative algebraic systems.

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