The Chitransh Academic & Research
An International Journal of Multidisciplinary Research
An Online Peer Reviewed / Refereed Journal
Volume1 | Issue 5 | December 2025
ISSN: 3049-303X (Online)
Website: www.thechitranshacadmic.in

A Brief Study About Flow Problem as a Constraint Problem of Viscous Fluids

Dr. Satyendra Narain Sharma and Dr. Ashutosh Mishra

ARTICLE DETAILS

Research Paper

Received: **16/11/2025**

Accepted: **21/12/2025**

Published: **31/12/2025**

Keywords: Flow problem, constraint problem, viscous fluids, Lagrange Multiplier Method, Penalty Function Method.

ABSTRACT

In this paper, we explore the formulation of the flow problem as a constraint problem in the context of viscous fluids. The study investigates how a constraint problem can be transformed into an unconstrained problem using two prominent methods: the Lagrange Multiplier Method and the Penalty Function Method. Additionally, an approach based on Lagrange's equation of motion, initially formulated by Eckert, is discussed. This approach helps in addressing the complexities of fluid flow by integrating constraints and optimizing the flow behavior. The paper aims to provide insights into how these methods contribute to solving flow problems in fluid dynamics, particularly when dealing with viscous fluids under constrained conditions.

1. Introduction:

The bilinear forms $B_t(w,v), B_v(w,v), \bar{B}_P(w,p)$ and $B_p(w_3,v)$ and the linear form $\ell(w)$ are defined by

$$\left.\begin{aligned} B_t(w,v) &= \int_{\Omega e} pW^T v dx \\ B_v(w,w) &= \int_{\Omega e} (Dw)^T C(DV)dx \\ \bar{B}_p(w,p) &= \int_{\Omega e} \left(D_1^T W\right)^T pdx \\ B_p(w_3,v) &= \int_{\Omega e} (w_3)^T \left(D_1^T V\right)dx \\ \ell(w) &= \int_{\Omega e} W^T f dx + \oint_{\Gamma_e} W^T t ds \end{aligned}\right\} \qquad \text{.........(2.1)}$$

where

$$D = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \qquad D_1 = \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}, \qquad c = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{......... (2.2)}$$

The transpose of a scalar (01 X01 matrix) used is the equation (2.1) may look, a bit strange at the moment but it is necessary to obtain the correct form of the finite element model. We will now formulate the finite element model in the following section.

2 . Formulation of the flow problem as a constraint problem:

The equations governing flows of viscous incompressible fluids can be viewed as equivalent to minimizing a quadratic functional with a constraint. Here we present the formulation, in the interest of simplicity, for the static case since the constraint condition does not involve time derivative terms. Then, we add the time derivative terms to study transient problems.

We begin with unconstrained problem described by the weak forms of the mixed model. Namely equation $B_t(w,v)+B_v(w,v)-\bar{B}_p(w,p)=\ell(w)-B_p(w_3,v)=0$ without the time derivative terms.

$$B_v(w,v)-\bar{B}_p(w,p)=\ell(w)-B_p(w_3,v)=0 \qquad \text{......... (3.1)}$$

where $B_v(w,v),\bar{B}_p(w,p),B_p(w_3,v)$ and $\ell(w)$ are defined in (2.1).

Now, suppose that the velocity field (v_x,v_y) is such that the continuity equation $\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0$ is satisfied identically. Then the weight function (w_1,w_2) being (virtual) variations of the velocity components, also satisfy the continuity equation.

$$\frac{\partial w_1}{\partial x}+\frac{\partial w_2}{\partial y}=0 \qquad \text{......... (3.2)}$$

As a result, variational problem (3.1) now can be stated as follows. Among all (v_x,v_y) that satisfy that the continuity equation (3.3), find the one that satisfies the variational problem.

$$B_v(w,v)=\ell(w) \qquad \text{......... (3.3)}$$

For all admissible weight functions (w_1,w_2), i.e. the one that satisfies condition (3.2).

3. Lagrange Multiplier Model

In the Lagrange multiplier method the constrained problem (3.6) is reformulated as one of finding the stationary points of the unconstrained functional.

The variational problem in equation (3.3) is a constrained variational problem because the solution (v_x, v_y) is constrained to satisfy the continuity equation $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$. We note that $B_v(w,v)$ is symmetric (because c is symmetric).

$$B_v(w,v) = B_v(v,w) \quad \text{......... (3.4)}$$

And it is linear is w as well as v, while $\ell(w)$ is linear in w. Hence, the quadratic functional is given by the expression.

$$I_v(v) = \frac{1}{2} B_v(v,v) - \ell(v) \quad \text{......... (3.5)}$$

Now, we can state that the equations governing steady flows of viscous incompressible fluids are the equivalent to minimize $I_u(v)$.

Subjected to the constrain

$$G(v) \equiv \frac{\delta v_x}{\delta x} + \frac{\delta v_y}{\delta y} = 0 \quad \text{......... (3.6)}$$

The constrained problem (3.6) can be reformulated as an unconstrained problem using the Lagrange multiplier method or the penalty function method. There are discussed next.

$$I_L(v,\lambda) \equiv I_v(v) + \int_{\Omega e} \lambda G(v)\,dxdy \qquad \text{......... (4.1)}$$

where $\lambda(x,y)$ is the Lagrange multiplier. The necessary condition for I_L to have stationary value is that

$$\delta I_L = \delta v_x I_L + \delta v_y I_L + \delta_\lambda I_L = 0 \rightarrow \delta v_x I_L = 0, \delta v_y I_L = 0, \delta_\lambda I_L = 0 \qquad \text{......... (4.2)}$$

where $\delta v_x, \delta v_y$ and δ_λ denote the partial variations with respect to v_x, v_y and λ, respectively. Calculating the first variation in (4.2), we obtain

$$0 = \int_{\Omega e}\left[\frac{\partial v_x}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}+\lambda\right)+\frac{\mu\sigma\partial v_x}{\partial y}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial y}\right)\right]dxdy - \int_{\Omega e}\partial v_x fxdxdy - \oint_{\Gamma e}\partial v_x txds \qquad \text{......... (4.3)}$$

$$0 = \int_{\Omega e}\left[\frac{\mu\sigma v_y}{\partial x}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+\frac{\partial\delta v_y}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y}+\lambda\right)\right]dxdy - \int_{\Omega e}\delta v_y fydxdy - \oint_{\Gamma e}\delta v_y t_y ds \qquad \text{......... (4.4)}$$

$$0 = \int_{\Omega e}\delta\lambda\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)dxdy \qquad \text{......... (4.5)}$$

where $t_x = \left(2\mu\frac{\partial v_x}{\partial x}+\lambda\right)n_x + \mu\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)n_y$

$$t_y = \mu\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)n_x + \left(2\mu\frac{\partial v_y}{\partial y}+\lambda\right)n_y \qquad \text{......... (4.6)}$$

or in vector from

$$B_v(w,v)+\bar{B}_p(w,\lambda)=\ell(w), B_p(\delta\lambda,v)=0 \qquad \text{.........} (4.7)$$

And the bilinear forms are the same as those in equation (2.1) and (2.2).

Comparison of equation (4.7) with equation

$$B_t(w,v)+B_v(w.v)-\bar{B}_p(w,p)=\ell(w)-B_p(w_3,v)=0$$

Reveals that $\lambda=-P$. Hence, the Lagrange multiplier formulation is the same as the velocity pressure formulation.

4. Penalty Model

In the penalty function method, the constrained problem (3.6) is reformulated as an unconstrained problem as following minimize the modified functional.

$$I_P(v)\equiv I_v(v)+\frac{\gamma^e}{2}\cdot\int_{\Omega x}[G(v)]^2\,dx \qquad \text{.........} (5.1)$$

where γe is called the penalty parameter. Note that the constraint is included in a least squares since into the functional. Seeking the minimum of the modified functional $I_P(v)$ is equivalent to seeking the minimum of both $I_v(v)$ and $G(v)$, the latter with respect to the weight γe. The larger the value of γe, the more exactly the constraint is satisfied.

The necessary condition for the minimum of I_p is

$$\delta I_p = 0 \qquad \text{......... (5.2)}$$

We have

$$0 = \int_{\Omega e}\left[2\mu\frac{\partial\delta v_x}{\partial x}\frac{\partial v_x}{\partial x} + \mu\frac{\partial\partial v_x}{\partial y}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) - \delta v_x fx\right]dxdy$$

$$-\oint_{\Gamma e}\delta v_x txds + \int_{\Omega e}\gamma e\frac{\partial\delta v_x}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) - dxdy \qquad \text{......... (5.3)}$$

$$0 = \int_{\Omega e}\left[2\mu\frac{\partial\delta v_y}{\partial y}\frac{\partial v_y}{\partial y} + \mu\frac{\partial\delta v_y}{\partial x}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) - \delta v_y fy\right]$$

$$-\oint_{\Gamma e}\delta v_y tyds + \int_{\Omega e}\gamma e\frac{\partial\delta v_y}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)dxdy \qquad \text{......... (5.4)}$$

or in vector form

$$B_p(w,v) = \ell(w) \qquad \text{......... (5.5)}$$

where ($w_1 = \delta v_x$ and $w_2 = \delta v_y$)

$$-\oint_{\Gamma e} \delta v_y t_y ds + \int_{\Omega e} \gamma e \frac{\partial \delta v_y}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) dxdy \qquad \text{.........} (5.4)$$

or in vector form

$$B_p(w,v) = \ell(w) \qquad \text{.........} (5.5)$$

where ($w_1 = \delta v_x$ and $w_2 = \delta v_y$)

$$B_p(w,v) = B_v(w,v) + \int_{\Omega e} \partial e\left(D_1^T w\right)^T D_1^T v dx$$

$$\ell(w) = \int_{\Omega e} w^T j dxdy + \oint_{\Gamma e} w^T t ds \qquad \text{.........} (5.6)$$

And $B_v(w,v)$ and D_1 are defined is equation (2.1) and (2.2). We note that the pressure does not appear explicitly in the weak form (5.3) and (5.4) although it is a part of the boundary stress $\lambda = -P$.

A comparison of the weak form in (5.3) and (5.4) with those in (4.3) and (4.4) show that

$$\lambda = \gamma e\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) = -P \text{ or } P = -\gamma e D_1^T v \qquad \text{.........} (5.7)$$

Where $v = v(\gamma e)$ is the solution of equation (5.3) and (5.4). Thus an approximation for the pressure can be post computed using (5.7).

The time derivative terms can be added to equation (4.3)–(4.5) as well as to (5.3) and (5.4) without affecting the above discussion. For the penalty mode, we have

$$0 = \int_{\Omega e}\left[P\delta v_x \frac{\partial v_x}{\partial t} + 2\mu \frac{\partial \delta v_x}{\partial x}\frac{\partial v_x}{\partial x} + \mu \frac{\partial \delta v_x}{\partial y}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \gamma e \frac{\partial \delta v_x}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right] dxdy$$

$$- \int_{\Omega e} \delta v_x fxdxdy - \int_{\Gamma e} \delta v_x txds \qquad \text{......... (5.8)}$$

$$0 = \int_{\Omega e}\left[P\delta v_y \frac{\partial v_y}{\partial t} + 2\mu \frac{\partial \delta v_y}{\partial y}\frac{\partial v_y}{\partial y} + \mu \frac{\partial \delta v_y}{\partial x}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \gamma e \frac{\partial \delta v_y}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right] dxdy$$

$$- \int_{\Omega e} fx\delta v_y dxdy - \oint_{\Gamma e} \delta v_y tyds \qquad \text{......... (5.9)}$$

$$\text{or, } B_t(w,v) + B_p(w,v) = \ell(w) \qquad \text{......... (5.10)}$$

where $B_t(w,v)$ is defined in equation (2.1) and $B_P(w,v)$ and $\ell(w)$ are defined in equation (5.6)

The penalty finite element model can be constructed using equation (5.8) and (5.9) [or equation (5.10)]

By substituting $\delta v_x = \psi_i$ and $\delta v_y = \psi_j$ and approximations

$$u_x(x,y,t) = \sum_{j=1}^{n} u_x^j(t)\psi_j(x,y)v_y(x,y,t)$$

$$= \sum_{j=1}^{n} v_y^j(t)\,\psi_j(x,y) \text{ for}$$

(v_x, v_y) we obtain

$$\begin{bmatrix} [M] & [O] \\ [O] & [M] \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix} + \begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = \begin{Bmatrix} F^1 \\ F^2 \end{Bmatrix} \qquad \text{……… (5.11)}$$

where $\left[K^{11}\right] = 2\left[S^{11}\right] + \left[S^{22}\right] + \left[S^{-12}\right]$

$$\left[K^{12}\right] = \left[S^{21}\right] + \left[S^{-21}\right]$$

$$\left[K^{22}\right] = \left[S^{11}\right] + 2\left[S^{22}\right] + \left[S^{-22}\right]$$

$$\left[K^{21}\right] = \left[S^{12}\right] + \left[S^{-21}\right]$$

With the coefficients

$$M_{ij} = \int_{\Omega e} P\psi_i^e \psi_j^e dxdy$$

$$S_{ij}^{\alpha\beta} = \int_{\Omega e} \mu \frac{\partial \psi_i^e}{\partial x_\alpha} \frac{\partial \psi_j^e}{\partial x_\beta} dxdy; \alpha.\beta = 1,2$$

$$S_{ij}^{\alpha o} = \int_{\Omega e} \mu \frac{\partial \psi_i^e}{\partial x_\alpha} \psi_j^e dxdy; d = 1,2$$

$$S_{ij}^{-\alpha\beta} = \int_{\Omega e} \gamma e \frac{\partial \psi_i^e}{\partial x_\alpha} \frac{\partial \psi_j^e}{\partial x_\beta} dxdy; \alpha.\beta = 1,2$$

$$S_{ij}^{-\alpha o} = \int_{\Omega e} \gamma \frac{\partial \psi_i^e}{\partial x_\alpha} \psi_j^e \, dxdy; \alpha = 1,2 \qquad \text{......... (5.12)}$$

$$F^1 = \int_{\Omega e} \psi_i^e fxdxdy + \oint_{\Gamma e} \psi_i^e txds$$

$$F^2 = \int_{\Omega e} \psi_i^e fydxdy + \oint_{\Gamma e} \psi_i^e tyds$$

In vector form, the finite element model is given by

$$M\Delta + (K_V + K_P)\Delta = f \qquad \text{......... (5.13)}$$

where (M, K_v and K_p are of the order $2n \times 2n$, and F is of the order $2n \times 1$)

$$M = \int_{\Omega e} P\psi^T \psi dx k_v = \int_{\Omega e} B_v^T C B_v dx$$

$$K_P = \int_{\Omega e} \gamma e B_P^T B_P dx \quad F = \int_{\Omega e} \psi^T pdx + \oint_{\Gamma e} \psi tdx \qquad \text{......... (5.14)}$$

$$B_v = D\psi, \qquad B_p = D_1^T \psi$$

For the unsteady case equation $M\Delta + K^{11}\Delta + K^{12}P = F^{1}, K^{21}\Delta = 0$ and (5.13) are further approximated using a time approximation scheme. Equations $M\Delta + K^{11}\Delta + K^{12}P = F^{1}, K^{21}\Delta = 0$ and (5.13) are of the form $M\Delta + K\Delta = F$ ....(5.15)

Where $\{\Delta\}$ denotes the vector of nodal velocities and pressure in the velocity-pressure formulation and only velocities in the penalty formulation. Using the α-family of approximation we reduce equation (5.13) (with $K = K_v + K_p$) to

$$\hat{K}\Delta_{S+1} = \tilde{K}\,\Delta s + \hat{F}_{S,S+1} \quad \text{.........(5.16)}$$

Where

$$\hat{K} = M + \alpha_1 K_{S+1}\tilde{K} = M - \alpha_2 K_S \quad \text{.........(5.17)}$$

$$\hat{F}_{S,S+1} = 0, F_{S+1} + a_2 F_5, a_1 = \alpha\Delta t, a_2 = (1-\alpha)\Delta t \quad \text{.........(5.18)}$$

where M and K for the penalty nodal are define in equation (5.14)

Reference

1. M. J. Crochet: *Numerical Simulation of Non-Newtonian Flow*, Elsevier, New York, U.K., 1984
2. P.M. Gresho, R.L. Lee, and R.L. Sant: *On the Time Dependent Solution of the Incompressible Navier-Stokes Equations in Two and Three Dimensions*, in *Recent Advances in Numerical Methods in Fluids*, Piheridge Press Limited, Swansea, 1980
3. K.A. Cliffe: *On Conservative Finite Element Formulations of the Inviscid Boussinesq Equation*, International Journal for Numerical Methods in Fluids, 1: 117-127, 1981
4. Ph.D. Thesis: *Application of F.E. Theory of Viscous Fluid Flow*, M.U., 1990
5. E.S. Kanazi, S. Allyn and Bacon Boston: *Principles of Fluid Dynamics*, 1962

6. Curle, N. and Davis, H.J.: *Modern Fluid Dynamics, Vol. I*, D. Van Nostrand Company Ltd., 1968
7. Aris, R.: *Vector Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, Englewood, New Jersey, 1962