



A STUDY OF BILATERAL GENERALIZED H-FUNCTION DISTRIBUTION

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ABSTRACT

In the present paper we introduce and study 'bilateral generalized H-function distribution' (BGHF distribution), which unifies almost all classical statistical distribution defined in doubly infinite range. The main contribution of this paper is its extraordinary flexibility and unification. We have shown that this new distribution is a super class for a very large set of familiar and useful probability distributions, such as the **Bilateral Bessel, Rice, Rayleigh, Chi, standard normal, and McKay distributions**, and so on, which occur as special cases by proper parameter choice.

INTRODUCTION

The H-function

The H-function occurring in this paper will be defined and represented in the following manner [1]:

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] = \frac{1}{2\pi\omega} \int_L \theta(\xi) z^\xi d\xi; \quad (\omega = \sqrt{-1}) \quad (1)$$

where

$$\theta(\xi) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \beta_j \xi) \prod_{j=n+1}^p \Gamma a_j - \alpha_j \xi} \quad (2)$$

m, n, p and q are non-negative integers satisfying $0 \leq n \leq p, 1 \leq m \leq q$; $\alpha_j (j = 1, 2, \dots, p)$ and $\beta_j (j = 1, 2, \dots, q)$ are assumed to be positive quantities for standardization purposes and an empty product in (2), if it occurs, is taken as unity. Further, the contour L runs from $-\omega\infty$ to $+\omega\infty$ such that the poles of $\Gamma(b_j - \beta_j \xi)$, $j = 1, 2, \dots, m$, lie to the right to L and the poles of $\Gamma(1 - a_j + \alpha_j \xi)$, $j = 1, 2, \dots, n$, lie to the left of L (see also [7]).

The contour integral given by (1) is absolutely convergent when the following conditions are satisfied [9, p.13, Eq.(2.2.11)]:

$$A = \sum_{j=1}^n \alpha_j - \sum_{j=n+1}^p \alpha_j + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j > 0 \text{ and } |\arg z| < \frac{1}{2} A\pi$$

Throughout this paper it is assumed that this function always satisfies the conditions given in [9, p.11].

Results

Main result

Bilateral	Generalized	H-function	distribution
$f(x) = K x ^{2\lambda-1} e^{-\gamma x^2} H_{p,q}^{m,n} \left[a x ^\mu \left \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right]; \quad -\infty < x < \infty \quad (3)$			

where

$$K^{-1} = \gamma^{-\lambda} H_{p+1,q}^{m,n+1} \left[a \gamma^{-\mu/2} \left| \begin{matrix} (1-\lambda, \mu/2), (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] \quad (4)$$

Here $H_{p+1,q}^{m,n+1}(ax^\mu)$ denotes the well-known Fox H-function [1].

Also

$$(i) \quad \mu > 0, \gamma > 0$$

$$(ii) \quad 2\lambda + \mu \min_{1 \leq j \leq m} \left(\frac{b_j}{\beta_j} \right) > 0 \quad (5)$$

(iii) The parameters involved in (3) are so restricted that $f(x)$ remains non negative and $\int_{-\infty}^{\infty} f(x) dx = 1$

Here we give a brief listing of some useful and well known probability density functions which follow as special cases of the bilateral generalized H-function (BGHF) distribution defined by (3).

Special cases of Bilateral generalized H-function Distribution

(i) Bilateral Bessel function distribution

On taking $a=1/4$, $\mu=2$, $m=2=q$, $n=0=p$, $\lambda=0$, $b_1=(b+v)/2$, $b_2=(b-v)/2$, $\beta_1=1=\beta_2$, the BGHF distribution reduces to a bilateral Bessel function distribution which is defined as follows

$$f(x) = K' \left| \frac{x}{2} \right|^{b-1} e^{-x^2} K_\nu(|x|) \quad ; \quad -\infty < x < \infty \quad (6)$$

where

$$(K')^{-1} = H_{1,2}^{2,1} \left[\frac{1}{4} \gamma^{-1} \left| \left(\frac{b \pm v}{2}, 1 \right) \right. \right] \quad (7)$$

and $\gamma > 0$, $|v| < b$

(ii) Bilateral type II Bessel function distribution

On taking $\mu=2$, $m=1$, $q=2$, $n=0=p$, $b_1=0$, $b_2=1-\lambda$, $\beta_1=1$, $\beta_2=1$ and replacing $\gamma \rightarrow \theta/2$ and $a \rightarrow -b^2/4$, the BGHF distribution reduces to a bilateral type II Bessel function distribution [8, p.352, Eq.(9.8.4)] which is defined as follows

$$f(x) = D |x|^\lambda e^{-(\theta/2)x^2} I_{\lambda-1}(b|x|) \quad ; \quad -\infty < x < \infty \quad (8)$$

where

$$D = \frac{\theta^\lambda}{2} \frac{e^{-b^2/2\theta}}{b^{\lambda-1}}, \quad (9)$$

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}$$
 is modified Bessel function of the first kind and $\theta > 0, \lambda > 0, b \geq 0$.

(iii) Rice distribution

In (8), if we take $\lambda=1$, $\theta = \frac{1}{\sigma^2}$ and $b = \frac{|\nu|}{\sigma^2}$, it reduces to Rice distribution as follows

$$f(x) = \frac{|x|}{2\sigma^2} e^{-\left(\frac{x^2+\nu^2}{2\sigma^2}\right)} I_0\left(\frac{|x\nu|}{\sigma^2}\right) \quad ; \quad -\infty < x < \infty \quad (10)$$

where $\sigma > 0$

(iv) Rayleigh distribution

Taking $\nu = 0$ in (10), we get Rayleigh distribution defined as follows

$$f(x) = \frac{|x|}{2\sigma^2} e^{-\left(\frac{x^2}{2\sigma^2}\right)} \quad ; \quad -\infty < x < \infty \quad (11)$$

where $\sigma > 0$

(v) Chi distribution

If we take $b=0, \theta = \theta/\sigma^2$ and $\lambda = \theta$ in (8), the bilateral type II Bessel function distribution reduces to chi distribution, which is given by

$$f(x) = \frac{1}{\Gamma\theta} \left(\frac{\theta}{2\sigma^2}\right)^\theta |x|^{2\theta-1} e^{-\left(\frac{\theta x^2}{2\sigma^2}\right)} \quad , -\infty < x < \infty \quad (12)$$

where $\sigma > 0$, θ is a positive integer.

(vi) Noncentral chi with two degree of freedom

On replacing $b \rightarrow \frac{b}{\nu^2}$, $\theta \rightarrow \frac{1}{\nu^2}$ and taking $\lambda=1$ in (8), it reduces to noncentral chi distribution with 2

degree of freedom, and shown as

$$f(x) = \frac{|x|}{2\nu^2} \exp\left(-\frac{b^2 + x^2}{2\nu^2}\right) I_0\left(\frac{b|x|}{\nu^2}\right) ; -\infty < x < \infty \quad (13)$$

where $\nu > 0, b \geq 0$.

(vii) Error function distribution

On taking $b = 0, \theta = 2h^2$ and $\lambda = 1/2$ in (8), it reduces to the well known error function distribution defined as follows

$$f(x) = \frac{h}{\sqrt{\pi}} \exp(-h^2 x^2) ; -\infty < x < \infty \quad (14)$$

where $0 < h < \infty$.

(viii) Standard normal distribution

If we take $b = 0, \theta = 1$ and $\lambda = 1/2$ in (8), it reduces to standard normal distribution given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) ; -\infty < x < \infty \quad (15)$$

(ix) Mckay distribution type II

On taking $m = q = 2, n = p = 0, a = 1/4, \mu = 4$ and replacing $x^2 \rightarrow \delta, \gamma \rightarrow c, \lambda - 1 \rightarrow a$ in (3), we get Mckay distribution type II (for $b=1$)

$$f(\delta) = \frac{|\delta|^a e^{-c\delta} K_a(|\delta|)}{\sqrt{\pi} 2^a \Gamma(a + 1/2)} (1 - c^2)^{a + \frac{1}{2}} \quad (16)$$

where $a > -1/2, |c| < 1, \delta \neq 0$.

(x) ${}_1F_1$ distribution

On taking $m = n = 1, p = 1, q = 2, a = -1, \mu = 2, \gamma = 1, a_1 = 1 - a, b_1 = 0, b_2 = 1 - b, \alpha_1 = \beta_1 = \beta_2 = 1$ in (3) and using Gauss summation theorem, we get the following presumably new distribution

$$f(x) = \frac{\Gamma(b-\lambda)\Gamma(b-a)}{\Gamma b \Gamma \lambda \Gamma(b-a-\lambda)} |x|^{2\lambda-1} e^{-x^2} {}_1F_1(a; b; x^2) \quad , -\infty < x < \infty \quad (17)$$

where $\lambda > 0$, $b-a-\lambda > 0$.

Now we study some important properties of our pdf $f(x)$ defined by (3).

Main result Second

1. The characteristic function

The characteristic function of $f(x)$ is given by

$$\phi(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} f(x) dx \quad (18)$$

where $i = \sqrt{-1}$ and $E(\cdot)$ stands for mathematical expectation and K is given by (4).

Substituting the value of $f(x)$ from (3), writing the H-function in series form [9, p.12, Eq.(2.2.4)], using a known result [3, p.342, Eq.(3.381.4)] to evaluate the integral and writing the result in terms of H-function of two variables [9, p.251], we obtain

$$\phi(t) = \frac{K}{2} \gamma^{-\lambda} \left\{ H_{1,0:p,q;0,1}^{0,1:m,n;1,0} \left[\begin{matrix} a\gamma^{-\mu/2} \left(1-\lambda, \mu/2, 1/2 \right) : (a_j, \alpha_j)_{1,p} ; - \\ \gamma^{-1/2} (it) \quad - : (b_j, \beta_j)_{1,q} ; (0,1) \end{matrix} \right] \right\} \quad (19)$$

Conclusion

Through this paper, we have been able to introduce and thoroughly examine a new and highly generalized probability model, the **Bilateral Generalized H-function (BGHF) distribution**. The main contribution of this paper is its extraordinary flexibility and unification. We have shown that this new distribution is a superclass for a very large set of familiar and useful probability distributions, such as the **Bilateral Bessel, Rice, Rayleigh, Chi, standard normal, and Mckay distributions**, and so on, which occur as special cases by proper parameter choice.



These findings present a comprehensive and resilient mathematical platform for dealing with the BGHF distribution. The main importance of this new distribution is that it can potentially describe complicated statistical behaviours that cannot be well-modelled by simpler, traditional distributions. Its capacity to consolidate disparate statistical models renders it an important theoretical tool for statisticians, physicists, and engineers.

The future research may be aimed at the implementation aspects of this distribution, for example, creating effective algorithms for **parameter estimation** from real data. Another promising direction would be generalizing this framework to a **multivariate BGHF distribution** and using it in modeling actual data sets of real-world applications from communications theory, signal processing, and econometrics to establish its practicality.

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